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NONLINEAR STRUCTURAL ANALYSIS
USING DYNAMIC RELAXATION METHOD
WITH IMPROVED CONVERGENCE RATE

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This paper develops the optimal time-step ratio and the critical damping for nonlinear structural analysis, when using dynamic relaxation method (DRM). The convergence rate of the DRM depends on the value of the time step and the critical damping factor. In the present study, the effect of the time-step ratio is separated from the time-step value. Additionally, the effect of the eigenvalue on the error function is investigated. Using the eigenvalue definition, the critical damping is updated in each dynamic relaxation (DR) step to decrease the required computations. Moreover, the optimal time-step ratio is calculated. It is also shown that the value of the constant time step has no effect on the rate of convergence. The proposed procedure provides a simple and accelerated DRM, which can be used in the linear and nonlinear analyses of structures. The efficiency of the new method was verified by its application to a wide range of typical structures. The results show that the suggested scheme accelerates the convergence rate of the DR process.

Nomenclature

$\mathbf{S}$ = Stiffness matrix

$\mathbf{P}$ = Force vector

$\mathbf{R}$ = Residual force vector

$\mathbf{M}$ = Mass matrix

$\mathbf{C}, c$ = Damping matrix and damping factor

$\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}$ = Displacement, velocity, and acceleration vectors

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$$\tau^k, h = \text{Time-step value}$$
$$\gamma = \text{Time-step ratio}$$
$$\boldsymbol{\varepsilon}^k = \text{Vector of errors in displacement}$$
$$\boldsymbol{\lambda}, \lambda_i = \text{Eigenvalue}$$

1. Introduction

Using a simple, efficient, and stable algorithm is very important in the nonlinear structural analysis. Explicit methods satisfy these needs better than the implicit ones. Furthermore, the explicit techniques require less memory and eliminate round-off errors. One of these tactics is the dynamic relaxation scheme, which converts a static problem to dynamic one by assuming mass and damping terms. Bearing this in mind, the low convergence rate is the most crucial difficulties in its application to nonlinear analysis, which has to be overcome.

The mathematical basis of the dynamic relaxation method (DRM) is rooted in the second-order Richardson method, which was presented by Frankel [1950]. Many researchers have used this method to analyze linear systems [Brew and Brotton (1971); Bunce (1972); Cassell and Hobbs (1976); Wood (1971)]. In the 1980s, Papadrakakis (1981), Underwood (1983), and Qiang (1988) extended the DRM formulation and proposed relations to calculate the DR parameters automatically. Subsequently, many papers have shown that the DRM is a powerful technique for solving engineering problems [Hegyi *et al.* (2006); Kadkhodayan and Zhang (1995); Kadkhodayan *et al.* (1997); Zhang *et al.* (1994); Zhang and Yu (1989)]. Munjiza *et al.* [1996, 1998] presented a clear description of damping cases. They considered the damping proportional to mass and stiffness and showed that, when the damping matrix is $2\mathbf{M}(\mathbf{M}^{-1}\mathbf{S})^{0.5}$, all modes are critically damped. Recently, Rezaiee *et al.* investigated the dynamic relaxation scheme and its numerical integration [Rezaiee-Pajand and Alamatian (2008a, b, c); Rezaiee-Pajand and Taghavian Hakkak (2006)]. Additionally, Kadkhodayan *et al.* [2008] suggested a modified time step.

A new and more general revision of the DR formulation is presented in this work. It is worth emphasizing that the time effects will be separated into the time step in the kth step and the time-step ratio. This formulation shows that the constant value of the time step does not affect the convergence rate. The mass factor is applied in a general form using the properties of diagonally dominant matrices. Consequently, the optimal time-step ratio is found, and the effects of the lowest and the highest eigenvalue are detected.

In addition, the critical damping for the DRM is evaluated. A simple iterative method is combined with the DR algorithm to calculate the critical damping. It will be shown that this new process converges much more quickly than the ordinary algorithm that uses the Rayleigh principle. An alternative formulation of the DR relations is established based on the idea of critical damping surfaces of Beskos and Boley (1980). Finally, some numerical examples are solved to show the capability of the formulation.

2. Mathematical Bases

Structural analysis requires solving a system of equations. The equations are linear or nonlinear and can be written in the following matrix form:

$$\mathbf{SX} = \mathbf{P}. \tag{1}$$

In the nonlinear systems, the stiffness matrix $\mathbf{S}$ and the force vector $\mathbf{P}$ are related to the displacement vector $\mathbf{X}$. The dynamic relaxation method solves Eq. (1) by transforming it into the dynamical space. To do so, the inertial and damping forces must be added to Eq. (1):

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{SX} = \mathbf{P}. \tag{2}$$

The velocity $\dot{\mathbf{X}}$ and the acceleration $\ddot{\mathbf{X}}$ are approximated for the DR method by a central finite difference about time t_k. Based on Fig. 1, one can write the following relations:

$$\dot{\mathbf{X}}^{k+1/2} = \frac{1}{\tau^{k+1}}(\mathbf{X}^{k+1} - \mathbf{X}^k)$$

$$\dot{\mathbf{X}}^k = \frac{1}{2}(\dot{\mathbf{X}}^{k-1/2} + \dot{\mathbf{X}}^{k+1/2}) \tag{3}$$

$$\ddot{\mathbf{X}}^k = \frac{1}{\tau^k}(\dot{\mathbf{X}}^{k+1/2} - \dot{\mathbf{X}}^{k-1/2}).$$

The mass matrix is usually diagonal and is related to the stiffness matrix and the time step τ^k. The damping matrix is defined by $\mathbf{C} = c\,\mathbf{M}$, in which, c is called damping factor. An iterative process is found by substituting Eq. (3) into Eq. (2):

$$\mathbf{X}^{k+1} = \mathbf{X}^k + \alpha\,\mathbf{M}^{-1}\mathbf{R}^k + \beta(\mathbf{X}^k - \mathbf{X}^{k-1}). \tag{4}$$

For convenience, the τ^k and the τ^{k+1}/τ^k ratio are written as h and γ, respectively. Other parameters are given below:

$$\alpha = \frac{2\gamma h^2}{2 + ch}, \quad \beta = \gamma\frac{2 - ch}{2 + ch}, \quad \mathbf{R}^k = \mathbf{P} - \mathbf{SX}^k.$$

The stability and the speed of the convergence rate of the process (4) are related to the values of α and β. Frankel [1950] gave some convergence conditions. He expanded the Richardson technique and called it "*the second-order Richardson*
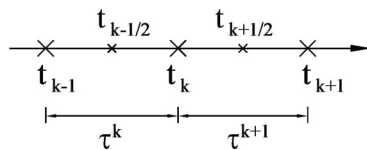

Fig. 1. The intervals of time in the finite difference.

method." This procedure can be achieved by using error function as $\boldsymbol{\varepsilon}^k = \mathbf{X}^k - \mathbf{X}^*$, in which $\mathbf{X}^*$ is the exact solution.

$$\boldsymbol{\varepsilon}^{k+1} = \boldsymbol{\varepsilon}^k - \alpha\mathbf{G}\boldsymbol{\varepsilon}^k + \beta(\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k-1}), \quad \mathbf{G} = \mathbf{M}^{-1}\mathbf{S}. \tag{5}$$

The $\mathbf{G}\,\boldsymbol{\varepsilon}^k$ term in the second-order Richardson scheme is a system of difference equations, whereas, in the DR method, the term is a system of linear equations. By defining the error eigenfunction, Richardson converted the system of difference equations into separate equations $\boldsymbol{\lambda}\boldsymbol{\varepsilon}^k$, where $\boldsymbol{\lambda}$ is a diagonal matrix and involves the eigenfunctions of $\mathbf{G}$. It is clear that $\boldsymbol{\lambda}$ is the eigenvalue matrix of $\mathbf{G}$ in the DR method. Consequently, Eq. (5) can be rewritten as:

$$\boldsymbol{\varepsilon}^{k+1} = [(1+\beta)\mathbf{I} - \alpha\boldsymbol{\lambda}]\boldsymbol{\varepsilon}^k - \beta\boldsymbol{\varepsilon}^{k-1}. \tag{6}$$

Frankel related $\boldsymbol{\varepsilon}^k$ and $\boldsymbol{\varepsilon}^{k+1}$ with the following linear equations:

$$\boldsymbol{\varepsilon}^{k+1} = \mathbf{E}(\alpha)\boldsymbol{\varepsilon}^k, \tag{7}$$

where $\mathbf{E}(\alpha)$ is the coefficient matrix. The maximum value of the error is related to the eigenvalues of the coefficient matrix. The eigenvalue matrix will be written as $\boldsymbol{\rho}$, and the following relations are derived:

$$\boldsymbol{\varepsilon}^{k+1} = \boldsymbol{\rho}\boldsymbol{\varepsilon}^k, \quad \boldsymbol{\varepsilon}^k = \boldsymbol{\rho}\boldsymbol{\varepsilon}^{k-1} \tag{8}$$

$$[\rho_i^2 - (1+\beta - \alpha\lambda_i)\rho_i + \beta]\left(\frac{\varepsilon_i^k}{\rho_i}\right) = 0. \tag{9}$$

It is important to note that the minimum value of ρ_i, which satisfies Eq. (9), will be achieved when the discriminant of the left term of Eq. (9) is zero. This leads to the following equations:

$$\Delta = (1+\beta - \alpha\lambda_i)^2 - 4\beta = 0 \quad \text{or} \quad 1+\beta - \alpha\lambda_i = \pm 2\sqrt{\beta}. \tag{10}$$

Frankel substituted λ_1 and λ_n into Eq. (10), when the condition $0 < \lambda_1 \le \lambda_i \le \lambda_n$ was satisfied.

$$1+\beta - \alpha\lambda_1 = +2\sqrt{\beta} \quad \text{and} \quad 1+\beta - \alpha\lambda_n = -2\sqrt{\beta}. \tag{11}$$

For the optimal numerical convergence, the values of α and β should satisfy Eq. (11). The following relations hold if the time step is constant:

$$\lambda_1 = \frac{2}{h^2} - \frac{2\sqrt{\frac{\beta}{\alpha^2}}}{h^2}, \quad \lambda_n = \frac{2}{h^2} + \frac{2\sqrt{\frac{\beta}{\alpha^2}}}{h^2}, \quad \beta + 1 = \frac{2\alpha}{h^2} \tag{12}$$

$$\rho_i = \frac{1}{2+ch}[(2 - h^2\lambda_i) \pm \sqrt{(2 - h^2\lambda_i)^2 - \beta(2+ch)^2}]. \tag{13}$$

3. Previous Works

To derive c and h, others have presented some conditions on Eq. (12). For convenience, only the best known relations will be presented in this section. Papadrakakis [1981] assumed that $h^2\lambda_1$ and $h^2\lambda_n$ must be symmetric about 2, because the roots of ρ are symmetric about $h^2\lambda_i = 2$ coordinates. He derived h^2 as follows:

$$h^2 = \frac{4}{\lambda_n + \lambda_1}. \tag{14}$$

If λ_1 in Eq. (12) is added to λ_n, then, Eq. (14) will result. On the other hand, multiplying these eigenvalues yields another relation, which Papadrakakis presented to calculate ch:

$$\lambda_n * \lambda_1 = \frac{4}{h^2}\left(1 - \frac{\beta}{\alpha^2}\right) = \frac{(ch)^2}{h^4} \quad \Rightarrow \quad ch = h^2\sqrt{\lambda_n * \lambda_1}. \tag{15}$$

The combination of Eqs. (14) and (15) yields:

$$ch = \frac{4\sqrt{\lambda_n * \lambda_1}}{\lambda_n + \lambda_1}. \tag{16}$$

Papadrakakis suggested that the mass matrix is equal to the factor of the diagonal entry of the stiffness matrix. Bunce [1972] and Cassell and Hobbs [1976] used Eqs. (14) and (16), such that their mass matrices differ from those of Papadrakakis. Underwood [1983] assumed $\lambda_1 \ll \lambda_n$ and wrote Eqs. (14) and (16) in the following approximate forms:

$$h^2 \leq \frac{4}{\lambda_n}, \quad c \approx 2\sqrt{\lambda_1}. \tag{17}$$

In the Underwood report, the mass is given as $m_{ii} = (h^2/4)\sum_j |s_{ij}|$. By utilizing $m_{ii} = \sum_j |s_{ij}|$, the maximum value of λ_n becomes one. This definition was applied by Qiang [1988] and the following relations were obtained:

$$h^2 = \frac{4}{1 + \lambda_1}, \quad ch = \frac{4\sqrt{\lambda_1}}{1 + \lambda_1}. \tag{18}$$

All relations presented to this end have the same nature, and the remaining point of Eqs. (14) and (16) is the effect of h. This paper will show later that the value of h has no effect on the convergence rate, when the time step is constant. In other words, the constant h acts only for the compatibility of the units in Eq. (2).

4. The Modified DR Formulation

To eliminate some of the drawbacks of DR, the effects of the time-step and the time-step ratio are separated in the proposed formulation and new definitions of the mass and damping matrices are given as below:

$$\mathbf{M} = h^2\mathbf{D}, \quad \mathbf{C} = hc\mathbf{D}, \tag{19}$$

where $\mathbf{D}$ is a diagonal matrix and a function of the stiffness matrix. Consequently, Eq. (4) can be rewritten as follows:

$$\mathbf{X}^{k+1} = \mathbf{X}^k + \gamma\{\alpha\mathbf{D}^{-1}\mathbf{R}^k + \beta(\mathbf{X}^k - \mathbf{X}^{k-1})\}, \quad \alpha = \frac{2}{2+c}, \quad \beta = \frac{2-c}{2+c}. \quad (20)$$

In this formulation, $\mathbf{G}$ is equal to $\mathbf{D}^{-1}\mathbf{S}$. Taking $\gamma = 1$, assuming that the time step is constant during the iterations, and taking into account the definitions of α and β, Eq. (9) can be written in the following form:

$$f(\rho_i) = \rho_i^2 - \alpha(2 - \lambda_i)\rho_i + \beta = 0$$

$$\xi_i = 2 - \lambda_i, \quad \Delta = \xi_i^2 - \frac{4\beta}{\alpha^2} = \xi_i^2 + c^2 - 4 \quad (21)$$

$$\rho_i = \frac{\alpha}{2}(\xi_i \pm \sqrt{\Delta}).$$

The convergence rate is dependent on the maximum value of $\|\rho\|$. The relation between the norm of ρ and the damping factor, c, is presented in Fig. 2.

The uppermost and undermost curves respectively belong to the eigenvalues farthest and nearest from 2. The convergence rate will increase if the largest difference between 2 and the eigenvalues decreases. If there is at least one $\lambda_i = 2$, the undermost curve will reach the $(2,0)$ point. This is the only curve that exists for the single DOF system. In other words, the SDOF system converges within the first iteration when c equals 2.

It is worth emphasizing that $2 - \lambda_1$ is greater than $\lambda_n - 2$. Consequently, the uppermost curve belongs to λ_1. Although Frankel suggested such a great identification, he assumed that λ_1 and λ_n are symmetric about 2, which is not always true. If they are symmetric about 2, then the curves related to λ_1 and λ_n will coincide. This case occurs just for the SDOF system. The uppermost curve shows the relation
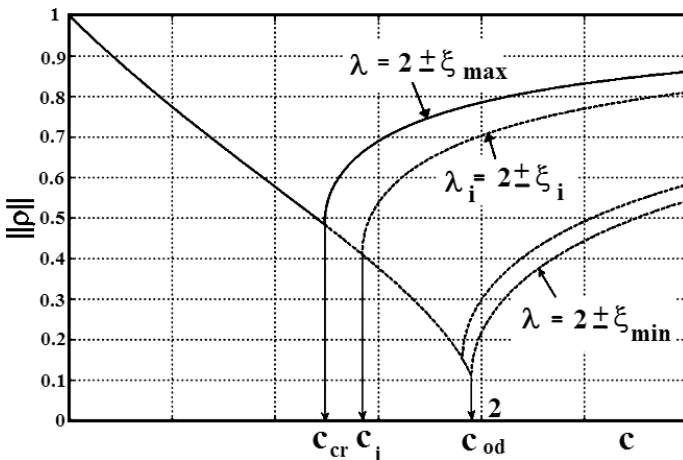

Fig. 2. $c - \|\rho\|$ curves.

between $\|\rho\|$ and c for a general structure. This function can be written as:

$$\|\rho\| = \begin{cases} \sqrt{\dfrac{2-c}{2+c}} & c \le c_{cr} \\[3mm] \dfrac{\xi_{\max} + \sqrt{\xi_{\max}^2 + c^2 - 4}}{2+c} & c \ge c_{cr} \end{cases} \tag{22}$$

$$c_{cr} = \sqrt{4\lambda_1 - \lambda_1^2},$$

where $\xi_{\max} = 2 - \lambda_1$. The proposed method uses Eq. (22) to calculate the critical damping. Further, the lowest eigenvalue is found via the power iteration scheme. The new DR algorithm is presented in Sec. 5. The fastest convergence rate will be obtained for the critical damping. Showing that, the number of the DR iterations is calculated using Eq. (22). If ρ will be constant through the DR process, one can write the following relation:

$$\varepsilon_i^j = \rho_i \varepsilon_i^{j-1}, \quad j = 1, \ldots, k$$

$$\|\boldsymbol{\varepsilon}^k\| = \|\rho\|^k \|\boldsymbol{\varepsilon}^0\| \quad \text{or} \quad \|\rho\|^k = \frac{\|\boldsymbol{\varepsilon}^k\|}{\|\boldsymbol{\varepsilon}^0\|} = \sqrt{\frac{(\mathbf{X}^k - \mathbf{X}^*)^T (\mathbf{X}^k - \mathbf{X}^*)}{(\mathbf{X}^0 - \mathbf{X}^*)^T (\mathbf{X}^0 - \mathbf{X}^*)}} = e^A \tag{23}$$

$$k = \frac{A}{\ln \|\rho\|}$$

Equation (23) gives the number of iterations, which is denoted by k. When the time step is constant, the number of iterations is a function of the allowable error, the initial error, the damping factor and the maximum value of $|2 - \lambda_i|$. The value of the time step has no effect in this case. However, it is important to note that the τ^{k+1}/τ^k ratio affects the total iteration numbers.

The discriminant of Eq. (21) is zero at the critical damping point. If the damping factor is less than the critical one, then the error factor ρ_i is a complex number. In such cases, the error factor changes with time, and the system becomes under damped. The error factor of the first mode will be real if the damping factor is greater than c_{cr}. It is possible to choose c_i (see Fig. 2) instead of c_{cr}, such that the ith mode converges critically. The modes with $\xi > \xi_i$ have a real-error factor and are over damped, whereas the other modes have complex ρ_j and are under damped. Because a structure's response is a combination of all modes, when $c < c_{cr}$, the response vector converges while under damped. The system will be over damped if the damping factor is greater than c_{od}. Although the damping state cannot be defined firmly for any chosen c_i, nevertheless, the first mode is governed the system, and it is usually over damped when $c > c_{cr}$. The first mode effect may occasionally be eliminated in some load cases, and the response of the structure converges critically using c_i. In this situation, the number of required iterations will increase if c_{cr} is used.

The diagonal entries of matrix $\mathbf{D}$ are also required to be defined. These values are obtained via Gerschgörin theorem and critical damping definition. Critical damping

is presented by the system's eigenvalues. On the other hand, the eigenvalues can be bounded using Gerschgörin rule. Combination of these two relations gives the artificial mass definition by Eq. (24). The more details are available in Appendix A.

$$d_{ii} = \frac{\zeta}{4} \sum_j |s_{ij}|, \quad \zeta = \frac{2s_{ii}}{\sum_j |s_{ij}|} \geq 1. \tag{24}$$

Bearing this in mind, Eq. (24) is more general than the previous similar ones. The common DRM, which was suggested by Underwood [1983], is obtained by utilizing $\zeta = 1.1^2$.

5. The Proposed DR Algorithm

The critical damping is related to the lowest eigenvalue of $\mathbf{G}$. It is usually calculated using the Rayleigh principle as $\lambda_1 = (\mathbf{X}^T \mathbf{S} \mathbf{X})/(\mathbf{X}^T \mathbf{D} \mathbf{X})$. Bunce [1972] noted that the Rayleigh quotient typically gives an upper bound for λ_1. This leads to over damping of the first frequency mode. Therefore, the response of the structure converges slowly. The simplest method for computing a better value for λ_1 is the power iteration given below [Krishnamoorthy and Sen (1991)]:

$$\mathbf{u}^1 = \{1, 1, 1, \ldots, 1\}^T$$
$$\begin{cases} \mathbf{v}^k = \mathbf{G} \, \mathbf{u}^k \\ \lambda^k = \max(\mathbf{v}^k) \\ \mathbf{u}^{k+1} = \mathbf{v}^k / \lambda^k \\ k = 1, 2, 3, \ldots \end{cases} \tag{25}$$

The result of the power iteration converges to the dominant eigenvalue, λ_n. However, it is possible to choose the shift $\mathbf{G} - a\,\mathbf{I}$, such as $a \geq \lambda_n$. With this shift, one can calculate the lowest eigenvalue when the process (25) converges to $\lambda_1 - a$. It should be noted that the upper bound of λ_n is 4, and it is not necessary to calculate "a." Furthermore, the convergence rate depends upon the λ_2/λ_1 ratio. The iteration method converges slowly if the first and second natural frequencies are close to each other. Nevertheless, from a practical viewpoint, the power iteration usually converges faster than the DR strategy. Therefore, this procedure can be combined with the DR iteration to obtain a new algorithm. The proposed scheme uses one step of the power method in each DR step to update the lowest eigenvalue. It should be noted, utilizing several steps of the power technique in each DR step improves the value of λ_1. However, this process increases the number of computations sharply and decreases the total number of the DR iterations slightly. In addition of using power iteration, the Rayleigh principle can be employed, too. Comparing two obtained values for the lowest eigenvalue and selecting the minor one leads to a better evaluation of critical damping. Furthermore, two parameters ζ and φ are inserted into the strategy to control the value of the artificial mass and damping. To this end, summing up all the propositions will lead to the suggested DR algorithm, which is presented in Appendix B.

It should be noted, comparing the outcomes of Rayleigh and power iteration method, step (e) of the procedure (56), is sometimes effective. In other words, step (d) of the power iteration usually gives a better evaluation of the lowest eigenvalue. In this case, step (e) can be removed from the proposed procedure (56) to reduce the computation efforts. It is reminded; there is no requirement to assemble the global stiffness matrix in the common dynamic relaxation method. Because, the product of $\mathbf{S}\,\mathbf{X}^k$ is used in the DR algorithm, and it is the vector of internal forces. This vector is assembled when the elemental stiffness matrices are calculated. Similarly, the outcome of $\mathbf{S}\,\mathbf{u}^k$ in the proposed tactic can be obtained without assembling the global stiffness matrix. Consequently, the required CPU times for evaluation the critical damping using the power iteration step and the Rayleigh principle are approximately the same. The suggested algorithm only requires additional memory storages to keep two vectors $\mathbf{S}\,\mathbf{u}^k$ and $\mathbf{u}^k$.

The time step in the proposed DR algorithm is constant and equal to 1. Utilizing this strategy, four dynamic relaxation procedures are on hand. In the first process, the parameters φ and ζ are taken zero and 1.1, respectively. In addition, the power iteration step is removed. Consequently, the common DRM is obtained, which is proposed by Underwood (1983). This scheme is called M1 in present article. The M2, M3, and M4 techniques use $\varphi = 1$ and ζ resulting from Eq. (24). In the M2 procedure, the lowest eigenvalue is evaluated using the Rayleigh quotient. On the other hand, the lowest eigenvalue of the M3 method is calculated by the power iteration. The M4 procedure utilizes all steps of the suggested process.

6. Alternative Formulation

The critical damping may also be found using an analytical technique. This is different from the common DR formulation, which uses some assumptions and reasoning with regards to the eigenvalues. Beskos and Boley (1980) presented the idea of critical damping surfaces. They provided two conditions to evaluate the critical damping of a general dynamical system. This idea is extended to find the critical damping of the DR procedure.

Some approximations are used in the common DRM. The first is the approximation of the derivatives of displacements using the finite difference method. The second is the linear assumption for error changes. Using these assumptions, Eq. (2) can be written as Eq. (26). For simplicity, when the time step is constant, one can write the following matrix equation:

$$\left[\mathbf{M}\frac{1}{h^2}(\boldsymbol{\rho} - 2\mathbf{I} + \boldsymbol{\rho}^{-1}) + \mathbf{C}\frac{1}{2h}(\boldsymbol{\rho} - \boldsymbol{\rho}^{-1}) + \mathbf{S}\right]\boldsymbol{\varepsilon}^k = \mathbf{0}. \tag{26}$$

This equation is similar to the one below, which governs the free oscillation of a dynamical system:

$$[\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{S}]\mathbf{X}^0 e^{\lambda t} = \mathbf{0}. \tag{27}$$

In other words, the DR system converts to a free oscillation one by transferring from the displacement space to the error space. The non-zero responses for Eqs. (26) and (27) can be achieved, when the determinate of the coefficients matrix will be zero.

$$P(\boldsymbol{\rho}) = \det\left[\mathbf{M}\frac{1}{h^2}(\boldsymbol{\rho} - 2\mathbf{I} + \boldsymbol{\rho}^{-1}) + \mathbf{C}\frac{1}{2h}(\boldsymbol{\rho} - \boldsymbol{\rho}^{-1}) + \mathbf{S}\right] = 0. \tag{28}$$

Assume that $\mathbf{M} = h^2\mathbf{D}$ and $\mathbf{G}\boldsymbol{\varepsilon}^k = \boldsymbol{\lambda}\boldsymbol{\varepsilon}^k$, Eq. (28) can be replaced by the following one:

$$P(\rho) = \det\left[\frac{(\rho - 1)^2}{\rho} + \mathbf{D}^{-1}\mathbf{C}\frac{\rho^2 - 1}{2\rho h} + \boldsymbol{\lambda}\right] = 0, \tag{29}$$

where ρ is one of the diagonal members of the matrix ρ. $P(\rho)$ is the characteristic polynomial of Eq. (26). When the structure is under damped, ρ_i is a complex number. If the system is over damped, then all components of ρ will be real numbers. Based on this discussion and the ideas set forth by Beskos, ρ is written in the following complex form:

$$\rho = \rho_r + id, \quad \rho_r, d \in R, \quad i = \sqrt{-1}. \tag{30}$$

One can apply the Taylor's series for $P(\rho)$, about the point $(\rho_r, d = 0)$:

$$P(\rho) = P(\rho_r, id) = P(\rho_r, 0) + id\left.\frac{\partial P}{\partial \rho}\right|_{(\rho_r,0)} + \frac{1}{2!}(-d^2)\left.\frac{\partial^2 P}{\partial \rho^2}\right|_{(\rho_r,0)}$$

$$+ \frac{1}{3!}(-id^3)\left.\frac{\partial^3 P}{\partial \rho^3}\right|_{(\rho_r,0)} + \cdots = 0. \tag{31}$$

If the real and imaginary parts are all together zero, then Eq. (31) is identically zero.

$$\begin{cases} P(\rho_r, 0) - \dfrac{d^2}{2!}\left.\dfrac{\partial^2 P}{\partial \rho^2}\right|_{(\rho_r,0)} + \cdots = 0 \\[3mm] \left.\dfrac{\partial P}{\partial \rho}\right|_{(\rho_r,0)} - \dfrac{d^2}{3!}\left.\dfrac{\partial^3 P}{\partial \rho^3}\right|_{(\rho_r,0)} + \cdots = 0. \end{cases} \tag{32}$$

In the critical and over damped cases, some eigenvalues or all of them are real numbers. Consequently, to achieve the critical damping, one can cancel the imaginary part of ρ. When $d = 0$, Eq. (32) is converted to the following form:

$$\begin{cases} P(\rho_r, 0) = 0 \\[3mm] \left.\dfrac{\partial P}{\partial \rho}\right|_{(\rho_r,0)} = 0. \end{cases} \tag{33}$$

In general, it is difficult to solve Eq. (33) because the damping matrix has to be constructed using q elements of c_j. In the q-dimensional c_j coordinates, Eq. (29) presents some curves for the critical or the over damped cases [Beskos and Boley (1980)]. On the other hand, it is easy to solve Eq. (32) in the DRM because the

damping matrix has one parameter, and it is related to the mass matrix. Using the definition (19), the characteristic polynomial can be written as follows:

$$P(\rho) = \det\left[\left(\frac{(\rho - 1)^2}{\rho} + c\frac{\rho^2 - 1}{2\rho}\right)\mathbf{I} + \boldsymbol{\lambda}\right] = 0. \tag{34}$$

Therefore, the necessary conditions for the critical damping are given below:

$$\begin{cases} \left|\left(\frac{(\rho - 1)^2}{\rho} + c\frac{\rho^2 - 1}{2\rho}\right)\mathbf{I} + \boldsymbol{\lambda}\right| = 0 \\ \frac{\partial}{\partial\rho}\left|\left(\frac{(\rho - 1)^2}{\rho} + c\frac{\rho^2 - 1}{2\rho}\right)\mathbf{I} + \boldsymbol{\lambda}\right| = 0. \end{cases} \tag{35}$$

The first relation in Eq. (35) is the same as the following one:

$$\frac{(\rho_i - 1)^2}{\rho_i} + c\frac{\rho_i^2 - 1}{2\rho_i} = -\lambda_i, \tag{36}$$

where λ_i is the ith eigenvalue of $\mathbf{G}$. The second relation of Eq. (35) means that the slope of the characteristic polynomial must be zero. In other words, at least one ρ_i must have a repeated root. Thus, in the critical damping, the discriminant of Eq. (36) is equal to zero, and c_{cr} is given by Eq. (22). Recalling that, one can substitute definition (30) into Eq. (36) and finds the same result:

$$\frac{(\rho_r + id - 1)^2}{\rho_r + id} + c\frac{(\rho_r + id)^2 - 1}{2(\rho_r + id)} = -\lambda_i. \tag{37}$$

Separating the real and the imaginary parts of Eq. (37), two relations are obtained as follows:

$$\begin{cases} \rho_r^3(2 + c) - 2\rho_r^2(2 - \lambda_i) + \rho_r[d^2(2 + c) + (2 - c)] - 2d^2(2 - \lambda_i) = 0 \\ \rho_r^2(2 + c) + d^2(2 + c) - (2 - c) = 0. \end{cases}$$

It is clear that d is zero in the critical damping case, and based on that, the following relations can be derived:

$$\begin{cases} \rho_{\mathrm{cri}}^2(2 + c) - 2\rho_{\mathrm{cri}}(2 - \lambda_i) + (2 - c_{\mathrm{cri}}) = 0 \\ \rho_{\mathrm{cri}}^2(2 + c_{\mathrm{cri}}) - (2 - c_{\mathrm{cri}}) = 0. \end{cases} \tag{38}$$

The maximum value of $\|\rho\|$ defines the convergence rate. This condition can be added to Eq. (38) as an objective function. As a result, the following nonlinear optimization program can be established:

$$\begin{cases} \max_i \rho_{\mathrm{cri}}^2 \\ \rho_{\mathrm{cri}}^2(2 + c) - 2\rho_{\mathrm{cri}}(2 - \lambda_i) + (2 - c_{\mathrm{cri}}) = 0 \\ \rho_{\mathrm{cri}}^2(2 + c_{\mathrm{cri}}) - (2 - c_{\mathrm{cri}}) = 0 \end{cases} \tag{39}$$

Specifically, Eq. (38) gives n damping cases with $(c_{\mathrm{cri}}, \rho_{\mathrm{cri}})$ coordinates. Among them, the critical case maximizes the value of the error. Therefore, if Eq. (39) is

solved simultaneously, the following relations result:

$$\begin{cases} c_{cr} = \sqrt{\lambda_1(4 - \lambda_1)} \\ \rho_{cr} = \dfrac{2 - c_{cr}}{2 - \lambda_1} = \sqrt{\dfrac{2 - c_{cr}}{2 + c_{cr}}}. \end{cases} \tag{40}$$

Once again, Eq. (22) is proven correct. The alternative formulation gives the same value for the critical damping as that using the common method.

7. The Time-Step Ratio

The modified DR method with a constant time step was presented in Sec. 4, and the effect of a variable time step will be evaluated here. For this purpose, taking into account Eq. (20), the error relation has the following form:

$$f(\rho_i) = \rho_i^2 + \alpha_\gamma(2 - \lambda_{\gamma i})\rho_i + \beta_\gamma = 0, \tag{41}$$

where new parameters are defined as below:

$$\alpha_\gamma = \frac{1 + \beta_\gamma}{2}, \quad \beta_\gamma = \gamma\beta, \quad \lambda_{\gamma i} = \gamma\frac{\alpha}{\alpha_\gamma}\lambda_i. \tag{42}$$

Based on the new method of Sec. 6, one can evaluate the critical damping by solving the following nonlinear programming:

$$\begin{cases} \min\left(\max_i \rho_i^2\right) \\ \rho_i^2(2 + c_i) - \rho_i[2\gamma(2 - \lambda_i) + (2 + c_i)(1 - \gamma)] + \gamma(2 - c_i) = 0 \\ \rho_i^2(2 + c_i) - \gamma(2 - c_i) = 0 \end{cases} \tag{43}$$

There are $i = 1, 2, \ldots, n$ damping cases for a specific value of γ. Among them, one case has the maximum value of $\|\rho\|$. It should be noted that comparing $\|\rho_1\|$ and $\|\rho_n\|$ usually gives $\max_i \rho_i^2$. If γ changes, the value of $\max_i \rho_i^2$ will change, too. The optimal time-step ratio will be in hand when $\max_i \rho_i^2$ reduces to a minimum. The error ratios and the damping factors at the extremum points are as below:

$$\rho_i = \frac{\alpha_\gamma}{2}(\xi_{\gamma i} \pm \sqrt{\Delta}), \quad \xi_{\gamma i} = 2 - \lambda_{\gamma i}, \quad \Delta = \xi_{\gamma i}^2 - 4\frac{\beta_\gamma}{\alpha_\gamma^2} \tag{44}$$

$$c_i = \frac{1}{(\gamma + 1)^2}\{4\gamma\sqrt{\lambda_i[2 + \gamma(2 - \lambda_i)]} - 2(\gamma - 1)(\gamma\lambda_i - \gamma - 1)\}. \tag{45}$$

Although Eqs. (21) and (44) are alike, it must be noted that $\lambda_{\gamma i}$ is a function of λ_i, γ and the damping factor. As a result, the norm of the error ratio here is more complicated than that in Sec. 6. Consequently, the effect of γ is shown via an example. The functions of $\|\rho\|$ for $\lambda_1 = 0.5, \lambda_n = 3, \gamma = 0.8, 1, 1.2$, and 1.4 are plotted in Fig. 3. According to Fig. 3, if γ increases, the curves related to λ_1 and λ_n will move down and up, respectively; i.e., when γ is less than 1, the curve corresponding to λ_1 moves up, and the minimum value of $\|\rho\|$ increases. As a result, the number of iterations will increase. By increasing γ from 1, the minimum value
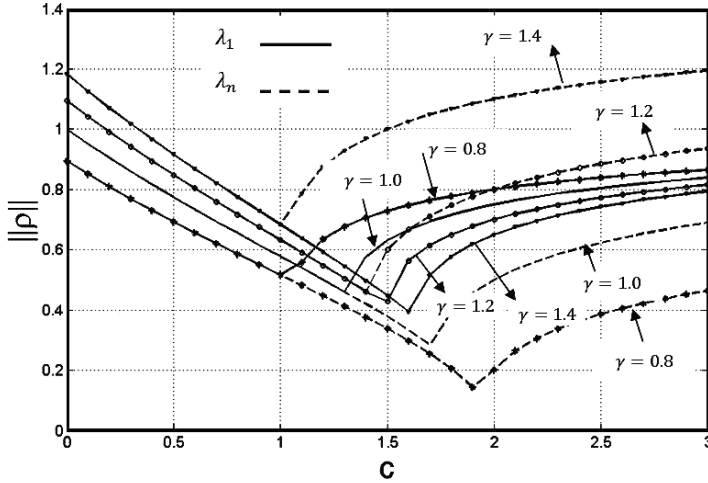
Fig. 3. $c - \|\rho\|$ curves for variable time step.

of $\|\rho_1\|$ decreases and $\min \|\rho_n\|$ increases. Moreover, if $\gamma \geq 1$ increases, then the number of required iterations will decrease until the minimum value of $\|\rho_1\|$ and $\|\rho_n\|$ will be the same. After that, λ_n is a determinative factor in the error ratio, and the rate of convergence decreases. It is clear that γ can vary until the minimum point of $\|\rho\|$ satisfies the condition of $\|\rho\| < 1$. Otherwise, the DR process will diverge. Based on this discussion, the optimal time-step ratio can be calculated when the damping ratios related to λ_1 and λ_n are the same.

$$c|_{\lambda=\lambda_1} = c|_{\lambda=\lambda_n}$$

$$\frac{\lambda_1}{\lambda_n} \sqrt{\frac{2(\gamma+1)}{\lambda_1} - \gamma} - \sqrt{\frac{2(\gamma+1)}{\lambda_n} - \gamma} = \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_n}\right)(1-\gamma). \tag{46}$$

In the SDOF system, for which the lowest and the highest eigenvalues are the same, Eq. (46) is satisfied for any value of γ. Using the formulation of Sec. 4, there is no requirement for iteration, and the DRM converges in the first step. As a result, the time-step ratio is meaningless for the SDOF system.

For multi-DOF systems, it is difficult to solve Eq. (46), and λ_n must be found. While later on, the effect of γ_{opt} on the iteration reduction will be given numerically, it is worth emphasizing, the required computations are very expensive and improvement of the total number of iterations is not considerable.

When $\lambda_n \gg \lambda_1$, using the Underwood time step ($\gamma_{\text{opt}}^2 = 4/\lambda_n$) gives good results, but is not the optimal value. It is remembered, Kadkhodayan *et al.* [2008] have tried to find the optimal time step numerically. They have suggested minimizing the square value of the residual force as below:

$$\text{RFF} = \sum_{i=1}^{ndof} (r_i^{k+1})^2 = \sum_{i=1}^{ndof} (r_i^k - \tau^{k+1} \dot{f}_i^{k+1})^2, \tag{47}$$

where $\dot{f}_i^{k+1}$ is the internal force increment of ith degree-of-freedom at the midpoint of the artificial time step. This parameter has the following relation:

$$\dot{f}_i^{k+1} = \sum_{i=1}^{ndof} s_{ij}\dot{x}_i^{k+1/2}. \tag{48}$$

It should be noted that the $(k + 1)$th time step is sometimes a very large or a very small number, and in this case, the process may be numerically unstable. Therefore, the time step should be limited by experimental values. If the time-step ratio is used, as proposed in the present article, this deficiency is eliminated. By using the formulation of Sec. 4, one can calculate the optimal time-step ratio in the kth step as follows:

$$\gamma^k = \frac{(\mathbf{A}^T\mathbf{R}^k)}{(\mathbf{A}^T\mathbf{A})}, \quad \mathbf{A} = \mathbf{S}(\alpha\mathbf{D}^{-1}\mathbf{R}^k + \beta\mathbf{\Delta X}^k). \tag{49}$$

Numerical experiences indicate that, if one uses a good evaluation of the critical damping suggested by procedure (56), the Kadkhodayan's scheme will not be very effective. Another way of finding the optimal τ^{k+1} was given by Alamatian (2007). He minimized the square of the residual energy by the following expression:

$$\Pi_r = \sum_{i=1}^{ndof} (\Delta x_i^{k+1} r_i^{k+1})^2 = (\tau^{k+1})^2 \sum_{i=1}^{ndof} [\dot{x}_i^{k+1/2}(r_i^k - \tau^{k+1}\dot{f}_i^{k+1})]^2. \tag{50}$$

To find the optimal time step from this function, minimal conditions must also be satisfied. This scheme improves the convergence rate within a few initial iterations.

8. Numerical Examples

Five structures have been chosen to be solved by the new algorithm to verify the validity and the merit of the proposed scheme numerically.

8.1. *Rod structure*

Figure 4 shows a single degree of freedom structure. The geometric properties of the structure are: $H = 1$ in $(0.0254\,\text{m})$ and $L_0 \cos\phi = 100$ in $(2.54\,\text{m})$. The value of section area and the modulus of elasticity for rod are $1\,\text{in}^2$ $(6.4516 * 10^{-4}\,\text{m}^2)$ and
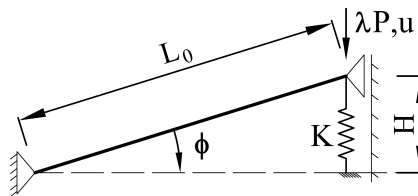

Fig. 4. TR1 structure.

10^7 lb/in^2 (68947.58 MPa), respectively. The stiffness of spring also is $K = 6$ lb/in (1.0508 KN/m). A vertical load, $P = 1$ lb (4.4497 N), is applied to the system, and then it increases using the load factor. The maximum value of λ is 24.

It should be noted, the **D** matrix in this example is equal to $S_{11}/2$ and the eigenvalue of **G** is 2. Based on these data, the critical damping for this structure is $c_{cr} = \sqrt{\lambda_1(4 - \lambda_1)} = 2$. It must be emphasized that c_{cr} is an exact value. Consequently, the proposed algorithm is converged in the first step for the linear analysis. To show the capability of the suggested tactic, a nonlinear analysis is also performed. Figure 5 illustrates the load–displacement curve of the structure. It is worth emphasizing, the results are in good agreement with the other researcher's findings [Crisfield (1997); Tatar (2002)]. According to Fig. 5, it can be seen that extreme nonlinear behavior can be easily traced with the proposed algorithm.

8.2. *Truss TR2*

In the most cases, the first mode governs the system responses. However, in some special cases, the effect of the first mode may be eliminated. In these cases, employing the relation (22) does not present the critical damping. The main aim of this example is studying one of these special cases. To fulfill the goal, the structure TR2, which is shown in Fig. 6, is considered. It is a 9-DOF truss. This structure is subjected to two kinds of loads. The two loading cases are shown in Figs. 6(c) and 6(d). The truss element properties are $A = 1, E = 10$, and $P = 0.1$.

TR2 is analyzed for different values of the damping factor. The final error of the residual force is 10^{-6}. The dots in Fig. 7 show the number of iterations for both load cases.

According to Fig. 7, each loading case has also its own critical damping. Furthermore, the relation (23) predicts the number of iterations for loading 2. However, this is unsuitable for loading 1. To find the reason behind this behavior, the relations between damping factor and $\|\rho\|$ for all modes of the dynamic relaxation system are shown in Fig. 8.
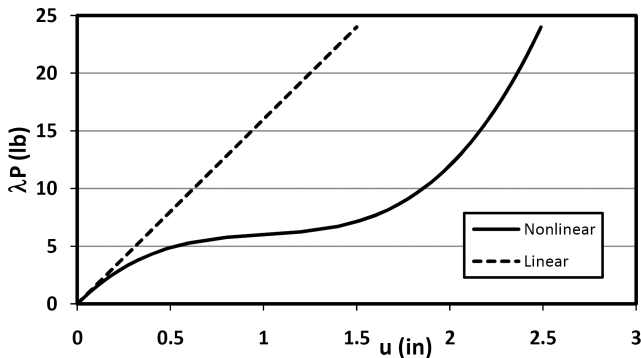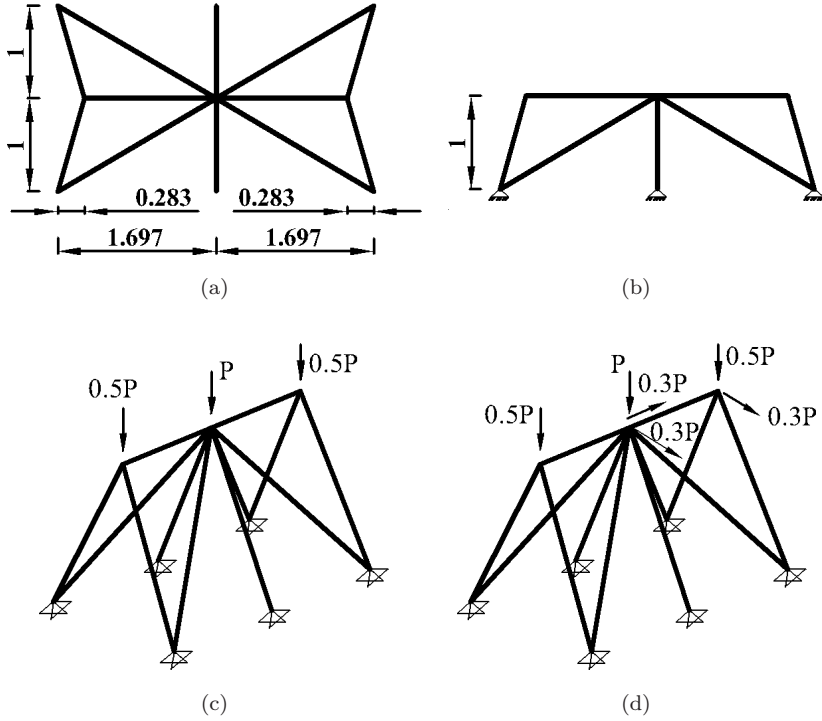

Fig. 5. The responses curves of TR1.

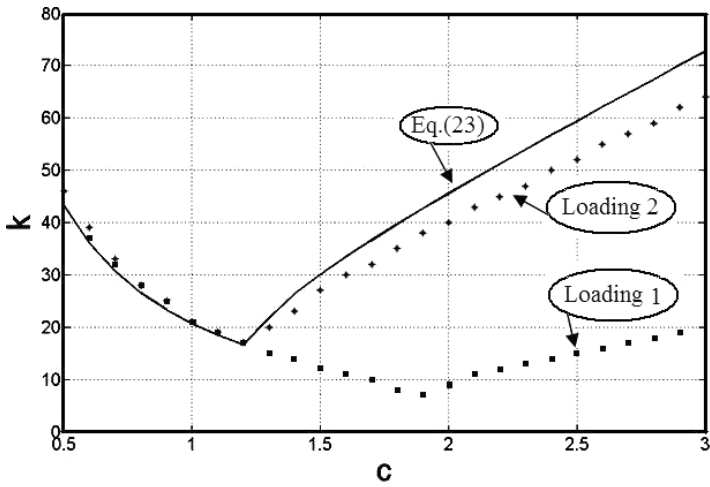Fig. 6. Truss TR2: (a) X–Y view, (b) X–Z view, (c) Loading 1 and (d) Loading 2.

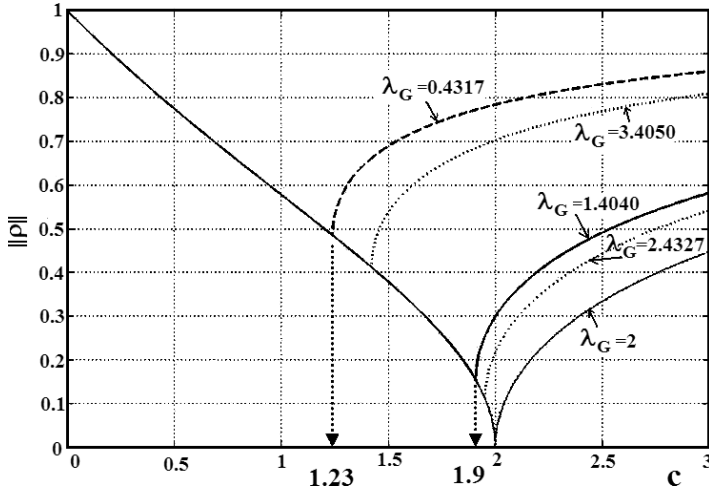Fig. 7. The number of iterations for TR2.

Fig. 8. The $c - \|\rho\|$ curves for TR2.

Figure 8 shows the corresponding critical damping with the first mode is 1.23. This value is the critical damping of loading 2. On the other hand, the critical damping factor of loading 1 is defined by the second mode and equals to 1.9. In particular, the loads in Fig. 6(c) are the special case for which $\mathbf{X}^0$ is a zero vector. Some displacements are zero (equal to the initial displacements) and, therefore, do not need any iteration. As a result, if a non-zero initial displacement vector is utilized, the critical damping factor 1.23 is found. To show this special result, the structure in Fig. 6(c) is analyzed with zero and identity initial displacement vectors. Figure 9 shows the required number of iterations for the first loading case
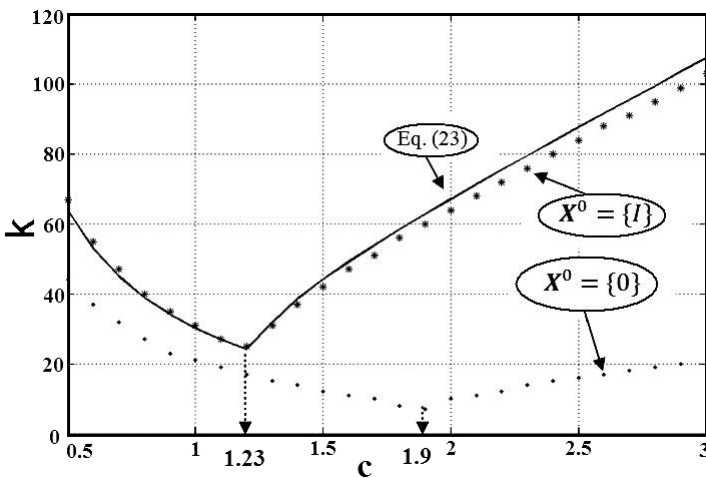

Fig. 9. The number of iterations for TR2 subject to loading 1.

and proves the expected topic. In other words, for most loading cases, Eq. (22) will give the correct value of the critical damping. It is interesting to note, using the Rayleigh principle, the convergence rate will increase in loading 1 analysis, because the displacement vector is swayed by the dominant mode.

8.3. *Tower truss*

The TR3 is a 48-DOF tower truss structure as shown in Fig. 10. The section modulus of elasticity for this tower is $2 * 10^{10}$. The areas of all sections are the same and equal to 0.01. The common DR method, presented in Eq. (4), is applied to solve this truss. For this structure, the effect of the time step will be studied.

   Three time steps, 0.5, 1, and 2, are chosen to run the DR process. The number of iterations and curves related to Eq. (23) are plotted on Fig. 11 for these time steps. When h is 1, the lowest and the highest eigenvalues of $\mathbf{G}$ are 0.0115 and 3.4137, respectively. In this case, the criterion (17) shows that $h \leq (2/\sqrt{\lambda_n}) = 1.0825$. Based upon this condition, it is not proper to choose $h = 2$. It is interesting to know, when h is 2, the analysis gives the same results. Because, the value of h does not affect the convergence rate, when the time-step ratio is 1.

8.4. *The dome truss*

The geometry and the loading of this structure are shown in Fig. 12. Its modulus of elasticity and the area of its sections are $2.1 * 10^4$ and 450, respectively.
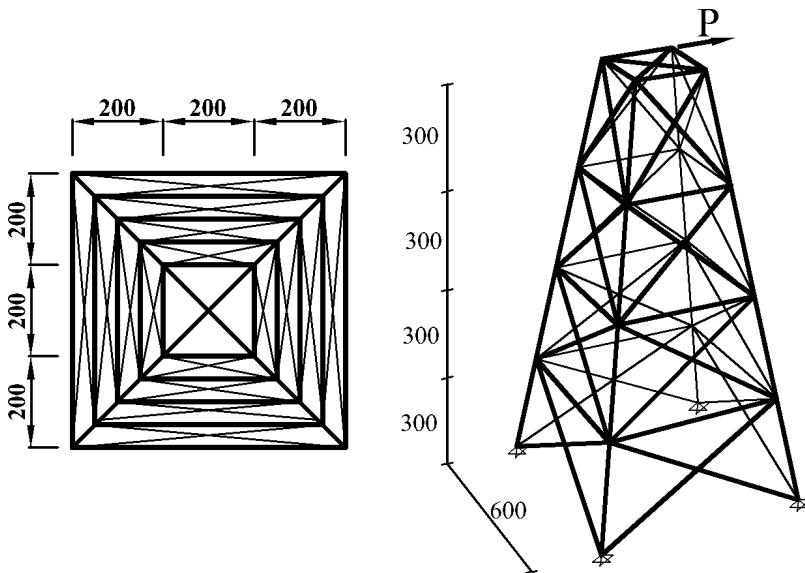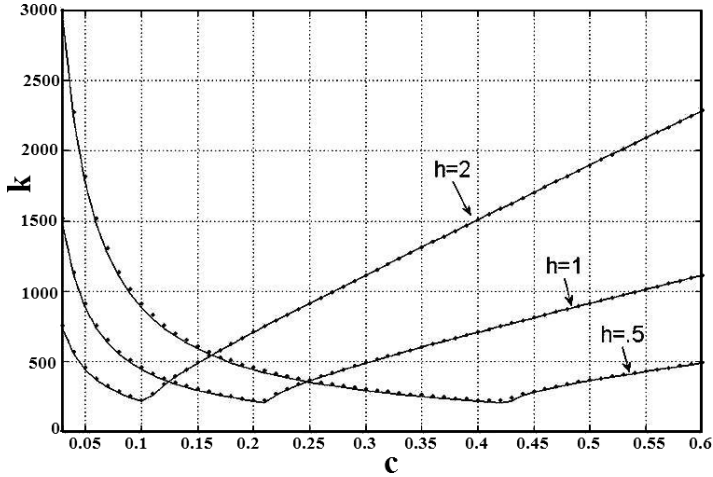

Fig. 10. The tower truss.

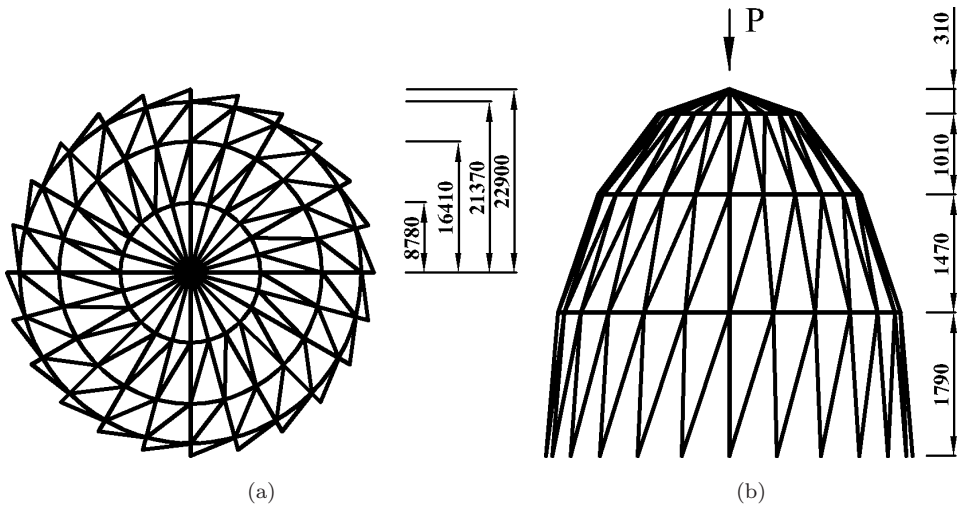Fig. 11. The number of iterations related to c based on Eq. (4).

Fig. 12. The TR4 structure: (a) X–Y view and (b) X–Z view.

When TR4 is analyzed by the proposed iterative procedure (56), the damping factor converges to $c = 0.14$, which is the critical damping factor. Showing this, the mentioned truss is analyzed using different values of damping factors. The number of required iterations corresponding with c is shown in Fig. 13(a). On the other hand, the critical damping based on the Rayleigh quotient and exact displacements becomes 0.26. In fact, about a 47% increase in the number of iterations will result from using the Rayleigh quotient. The convergence curves of $\|\mathbf{X}^k\|$ for $c = 0.05$, 0.14, and 0.26 are plotted in Fig. 13(b). These curves show the under-damping, critical, and over-damping cases.
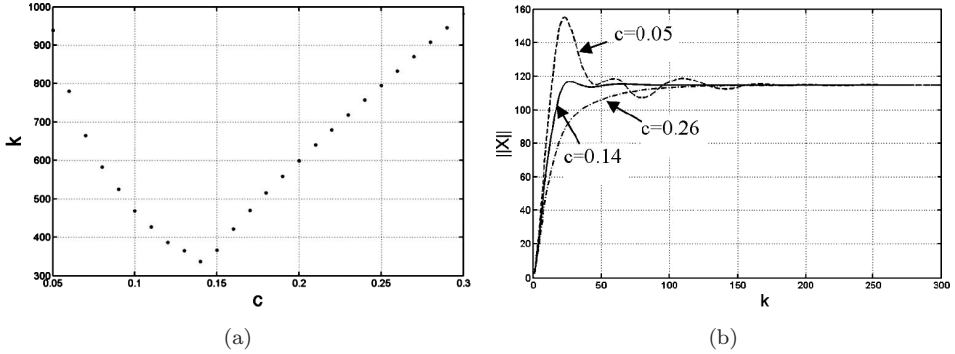
(a)   (b)

Fig. 13. Analysis of the dome truss: (a) The c–k curve and (b) convergence curves.
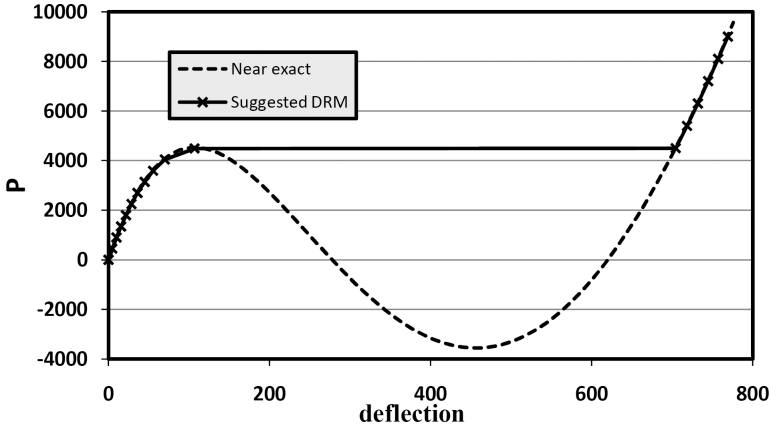

Fig. 14. Nonlinear response of the dome truss.

This dome structure has a vicious nonlinear behavior. The suggested dynamic relaxation technique can trace this behavior until the load limit state. Figure 14 shows the deflection of top-point of the structure when the maximum value of load is 9000. The aforementioned figure also shows the near exact solution that is found by employing a displacement control method and a fine incremental deflection. It is worth emphasizing, the jump in the response curve is rooted in the incremental load method. The process of finding the lowest eigenvalue does not affect this phenomenon. In other words, the proposed dynamic relaxation procedure can find the DR parameters even in the case of the structural stiffness changes rapidly.

8.5. *Building frame*

A six-story moment resistance frame is analyzed in this example. Two linear and nonlinear behaviors will be studied. Figure 15 shows the geometry of this structure and its loading. The section properties are presented in Table 1.
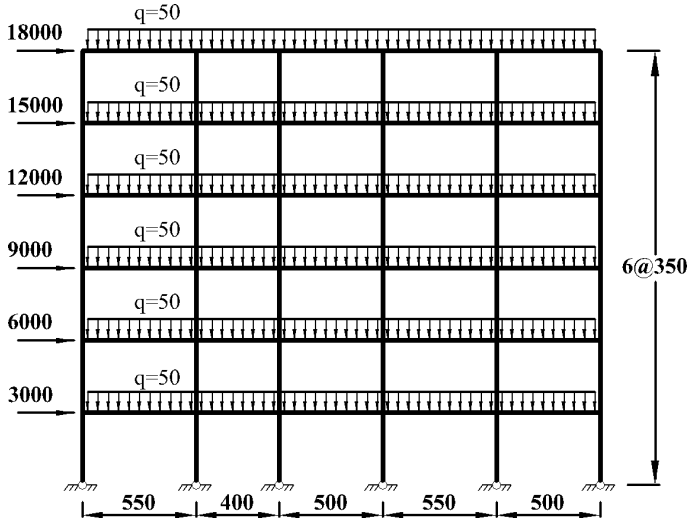
Fig. 15. Frame structure.

Table 1. The properties of sections and material.

Section	Member	Story	A	I	E
C1	Column	1, 2, 3	76.13	25470	$2 * 10^6$
C2	Column	4, 5, 6	66.45	21230	
B	Beam	All	58.84	15610	

The p–δ curve for the horizontal displacement of the sixth story is plotted in Fig. 16. Alamatian (2007) analyzed this structure, and the results for linear and nonlinear solutions shown by Fig. 16 are the same as those obtained by Alamatian. Analyzing this frame with nonlinear behavior clearly indicates that the ability of the proposed technique can give the critical damping, even when the stiffness changes.

8.6. *Comparison study*

A comparison of the proposed process and the common DR method is presented for both the linear and nonlinear analyses of the aforementioned structures. TR2 is analyzed subjected to loading case 1. In the nonlinear analysis, the load step is 0.1 of the final value. The time step is constant and equal to 1, and the admissible error for the residual force is 10^{-4}. The total numbers of iterations of the common DRM are presented in Table 2 by k_{M1}. The percentage of the reduced iterations of method M2 respect to the M1 process is calculated. The techniques M3 and M4 are also compared with M2 and M3 procedures, respectively. The percentages of the reduced CPU time of running the new methods are also calculated in the similar manner. All outcomes are inserted in Table 2.
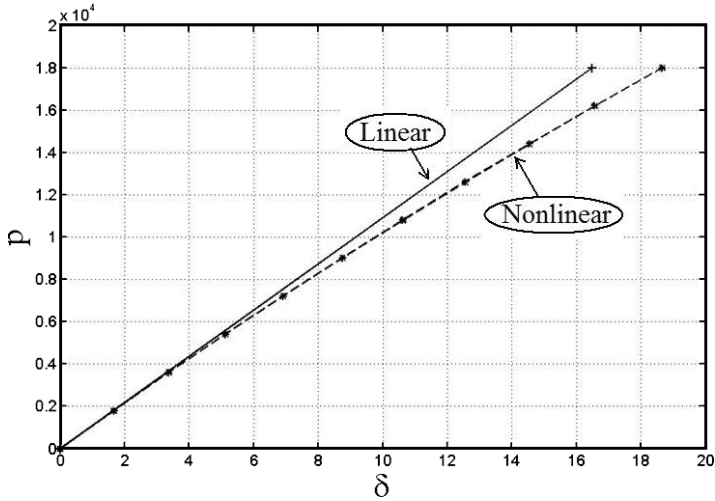
Fig. 16. The p–δ curve.

Comparing the M2 and M1 methods shows that using the mass definition of Eq. (24) decreases the number of iterations in the small structures. However, this process does not have too much effect for the large systems. On the other hand, studying the outcomes of methods M3 and M4 confirm the advantage of M3, especially for large structure. In other words, employing the power iteration step in M3 tactic instead of the Rayleigh principle in the M2 process reduces the total number of iterations too much. The system TR2 subject to loading 1 is a special case that the first mode is not effective in the structural response. As it is discussed in Sec. 8.2, using the Rayleigh principle is better than the power tactic in this system. It is reminded, the method M4 compare the value of Rayleigh's ratio and the result of power iteration step and utilizes the minor one. In the case of the critical damping of M2 is greater than M3, utilizing the strategy M4 is not also efficient in the analysis TR2. Table 2 shows that, in comparison with M3, utilizing the process M4 does not always decrease the total number of iterations.

The same conclusion can be obtained by studying the reduced duration time when the M2, M3, and M4 methods are utilized. It should be noted that the analysis time of a linear small structure is very little and sensitive. In general, the nonlinear analyses require appreciable times to run. Consequently, a deduction based on the CPU time of the nonlinear analyses is more valuable. Comparing the outcomes of nonlinear analyses show that required time is approximately proportional to the utilized iterations. As a result, using a better evaluation of the critical damping for the large structures appreciably decreases the time and the iterations necessary for convergence. The combination of the iterative procedure and the Rayleigh method, in method M4, can sometimes help the analyzer to achieve this goal.

Table 2 shows that employing the tactic M4 for the mentioned frame structure causes a considerable improvement. To find the reason behind this fact, the

Table 2. The effect of methods on the total number of iterations and CPU time.

| Structure	DOFs	Behavior	No. Iterations k_{M1}	Reduced Iterations $(M_i, M_j) = (k_{M i} - k_{M j})/k_{M j}$			Reduced CPU Time $(M_i, M_j) = (t_{M i} - t_{M j})/t_{M j}$		
				(M2, M1)	(M3, M2)	(M4, M3)	(M2, M1)	(M3, M2)	(M4, M3)
TR2	9	Linear	36	80.6%	−42.9%	0.0%	21.5%	−1.6%	1.6%
		Nonlinear	240	79.2%	−60.0%	0.0%	79.0%	−59.8%	0.0%
TR3	48	Linear	257	5.1%	11.1%	0.0%	12.0%	−13.6%	12.0%
		Nonlinear	2139	5.1%	15.1%	0.0%	4.4%	14.8%	0.1%
TR4	219	Linear	694	4.9%	52.6%	0.0%	4.1%	18.6%	−0.1%
		Nonlinear	5862	4.5%	36.9%	2.5%	5.4%	33.9%	0.5%
Frame structure	114	Linear	2574	4.3%	17.5%	24.9%	4.6%	16.8%	24.9%
		Nonlinear	16950	4.0%	16.4%	6.0%	7.9%	12.1%	6.3%
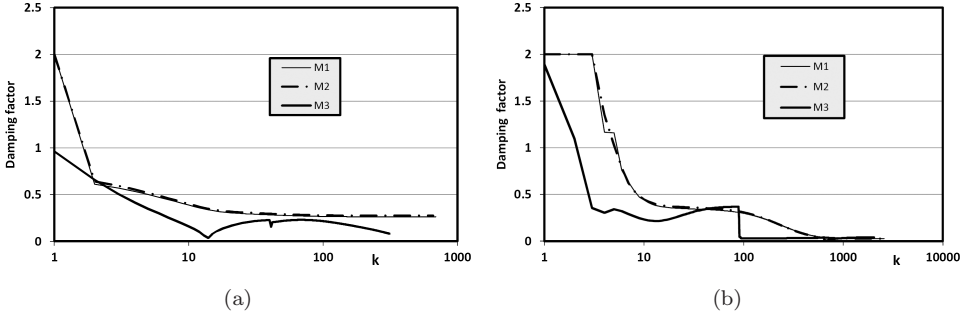
Fig. 17. The curves of damping factors: (a) dome truss and (b) frame structure.

variations of damping factor over the DRM iterations are shown in Fig. 17 for frame and also dome structures. Three tactics M1, M2, and M3 are considered. As it is shown in the mentioned figure, the optimized mass in the M2 tactic, instead of M1, has no appreciable effect on the critical damping for these structures. On the other hand, using the power iteration step for the M3 method presents a better evaluation of the critical damping. Figure 17 also shows that the result of the Rayleigh quotient is less than the outcome of the power tactic for a few steps. These steps for the foregoing frame system are more than the dome structure. In other words, the effectiveness of the procedure M4 is different from one structure to another one. Although, comparing the results of the Rayleigh quotient and the power tactic is not usually effective but its elimination does not decrease the CPU time.

8.7. The effect of the time-step ratio

In order to study the effect of the time-step ratio, the TR2 to TR4 trusses are analyzed. The exact values of λ_1 and λ_n are used to calculate γ, and the critical damping. These values are presented in Table 3. The number of iteration versus values of γ is also given in Table 3. The results show that the required computations are very expensive and also the analysis improvement is not significant.

In the following, the effect of the time-step ratio is studied for the TR4 structure. The dome truss is analyzed based on the formulation of Sec. 7, and the curves of $\|\mathbf{X}^k\|$ are plotted for different values of γ in Fig. 18. The value of the error ratio is

Table 3. The number of iterations for TR2 to TR4.

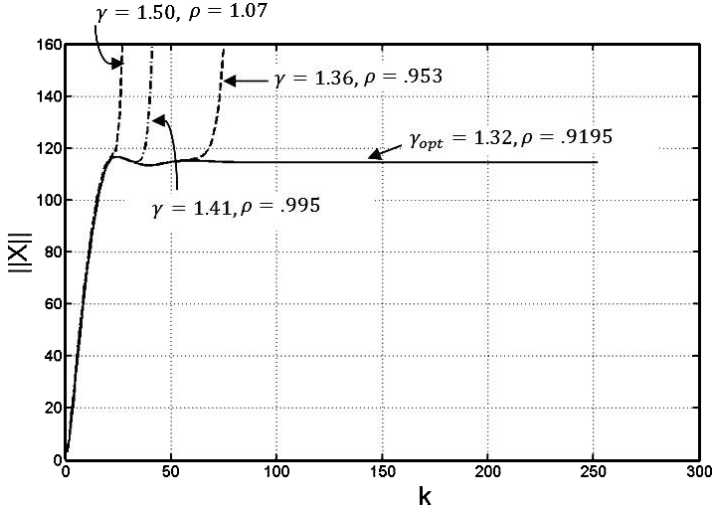|						No. of Iterations		
Truss	λ_1	λ_n	γ_{opt} Eq. (45)	$c_{cr}|\gamma = 1$	$c_{cr}|\gamma_{\mathrm{opt}}$	$\gamma=1$	γ_{opt} Eq.(45)	$\gamma_{\mathrm{opt}} = \sqrt{\frac{4}{\lambda_n}}$
TR2	0.4317	3.4050	1.05	1.24	1.29	9	9	9
TR3	0.115	3.4137	1.30	0.214	0.485	191	181	183
TR4	0.0049	3.3941	1.32	0.14	0.42	275	252	260

Fig. 18. The effect of γ on convergence.

calculated from Eq. (44), and is shown in this figure. The critical time-step ratio for this structure is approximately 1.41. The value of ρ is 0.93, when the value of the time step is constant. It is clear that the values of $\gamma \geq \gamma_{cr}$ cause diverging. If one uses the time-step ratio very near to the critical value, the numerical process will diverge. As a result, to solve a system with variable time step, the analyzer should calculate the exact value for γ_{opt} or an approximated value that is less than γ_{opt}. The reason is that the optimal time-step ratio becomes close to the critical one.

9. Conclusions

In this paper, the effect of time is divided into two components. These parts are the kth time step and the time-step ratio. It is shown that the value of the constant time step does not affect the convergence rate. On the other hand, the optimal time-step ratio is less than 2 and greater than or equal to 1, and within these bounds, there is a critical value. If one chooses a time-step ratio very close to the critical one or greater than it, the process will diverge. The time-step ratio less than 1 does not cause divergence. As discussed earlier, it decreases the convergence rate. When the time-step ratio is less than the optimal one, the lowest eigenvalue will govern the convergence rate. For the optimal time-step ratio, the effects of the lowest and the highest eigenvalue in the error function are the same. Aside from these cases, the convergence rate will be affected by the highest eigenvalue.

In addition, the critical damping for the DRM is computed and is related to the lowest eigenvalue, when the time step is constant. One step of the power method is combined with the DR algorithm to evaluate the critical damping. This new process converges much faster than the ordinary process that uses the Rayleigh principle.

If one applies a good measure of the critical damping, using a variable time step is not required, because the optimal time-step ratio is close to the critical one and the required computation for the optimal value is expensive. In other words, the optimal time-step ratio will not considerably reduce the total number of iterations.

Appendices

Appendix A: Finding the artificial masses

The critical damping equation is a half-circle in the λ–c coordinates. The center of the circle is located at the $(2,0)$ point with a radius of 2. The damping factor is not zero for the DR procedure, and as a result, all eigenvalues satisfy the $0 < \lambda < 4$ condition. It must be noted that $2 - \lambda_1$ is greater than $\lambda_n - 2$. This simplicity gives the following condition:

$$\lambda_1 + \lambda_n \leq 4. \tag{51}$$

If λ_n governs the error, the condition $\lambda_1 + \lambda_n \geq 4$ should be satisfied. This condition is in conflict with $0 < \lambda < 4$ condition. Therefore, the lowest eigenvalue governs the error when the time step is constant. On the other hand, the condition $0 < \lambda_1 + \lambda_n \leq 4$ satisfies the convergence of the DR process. This condition shows that λ_n must be less than 4. It is possible to calculate the upper bound of the highest eigenvalue by Gerschgörin theorem as shown follows:

$$|\lambda_i - g_{ii}| \leq \sum_{j \neq i} |g_{ij}| \quad \Rightarrow \quad \lambda_n \leq \sum_j |g_{ij}| = \frac{1}{d_{ii}} \sum_j |s_{ij}|. \tag{52}$$

Based on this equation and the $\lambda_n < 4$ condition, d_{ii} can be expressed in one of the following forms:

$$d_{ii} > \frac{1}{4} \sum_j |s_{ij}| \quad \text{or} \quad d_{ii} \geq \frac{1}{4} \sum_j |s_{ij}|. \tag{53}$$

Using the equality sometimes satisfies the convergence condition. However, the inequality condition is occasionally an urgent need. Researchers have typically suggested a factor greater than one. For example, Underwood (1983) proposed a factor equal to 1.1^2 to ensure stability. Alamatian (2002) found that this factor can be changed from one structure to another. Another condition can also be constituted when **S** is a diagonally dominant matrix. The **S** is diagonally dominant if the following conditions are held:

$$|s_{ii}| > \sum_{j \neq i} |s_{ij}| \quad \text{or} \quad 2s_{ii} > \sum_j |s_{ij}|. \tag{54}$$

By combining Eqs. (54) and (52), the following relation for the diagonally dominant matrices can be found:

$$d_{ii} = \frac{s_{ii}}{2}. \tag{55}$$

It should be reminded that Alamatian (2007) gave this definition by another formulation. Based on this discussion, d_{ii} will be expressed in a general form as follows:

$$d_{ii} = \frac{\zeta}{4} \sum_j |s_{ij}|, \quad \zeta = \frac{2s_{ii}}{\sum_j |s_{ij}|} \geq 1.$$

Appendix B: The suggested DR algorithm

(a) Assume $k = 0, \mathbf{X}^k = \mathbf{\Delta X}^k = \mathbf{0}$ and $\mathbf{u}^k = \{1, 1, 1, \ldots, 1\}^T$

(b) Compute $\mathbf{S} \mathbf{X}^k, \mathbf{S} \mathbf{u}^k, d_{ii} = \frac{\zeta}{4} \sum_j |s_{ij}|$, and $\mathbf{R}^k = \mathbf{P} - \mathbf{S} \mathbf{X}^k$

(c) Calculate $\mathbf{R}^k = \mathbf{P} - \mathbf{S} \mathbf{X}^k$, if $\|\mathbf{R}^k\| \leq \varepsilon_r$ then STOP

(d) $\mathbf{v}^k = [\mathbf{G} - a\mathbf{I}]\mathbf{u}^k = \mathbf{D}^{-1}\mathbf{S} \mathbf{u}^k - a\mathbf{u}^k, \lambda^k = \max(\mathbf{v}^k), \mathbf{u}^{k+1} = \mathbf{v}^k/\lambda^k, \lambda_1^k = \lambda^k + a$

(e) $\lambda_R^k = \dfrac{\mathbf{X}^{k^T}\mathbf{S} \mathbf{X}^k}{\mathbf{X}^{k^T}\mathbf{D} \mathbf{X}^k}$, if $\lambda_R^k < \lambda_1^k \Rightarrow \lambda_1^k = \lambda_R^k$ $\qquad\qquad$ (56)

(f) $c^k = \sqrt{4\lambda_1^k - \varphi(\lambda_1^k)^2}$

(g) $\mathbf{\Delta X}^{k+1} = \alpha\mathbf{D}^{-1}\mathbf{R}^k + \beta\mathbf{\Delta X}^k, \alpha = \dfrac{2}{2 + c^k}, \beta = \dfrac{2 - c^k}{2 + c^k}$

(h) $\mathbf{X}^{k+1} = \mathbf{X}^k + \mathbf{\Delta X}^{k+1}$

(i) if $\|\mathbf{\Delta X}^{k+1}\| \leq \varepsilon_x$ then STOP

(j) $k = k + 1$, GOTO b

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